Statistics of the Occupation Time of Renewal Processes

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We present a systematic study of the statistics of the occupation time and related random variables for stochastic processes with independent intervals of time. According to the nature of the distribution of time intervals, the probability density functions of these random variables have very different scalings in time. We analyze successively the cases where this distribution is narrow, where it is broad with index $\theta < 1$, and finally where it is broad with index $1 < \theta < 2$. The methods introduced in this work provide a basis for the investigation of the statistics of the occupation time of more complex stochastic processes (see joint paper by G. De Smedt, C. Godrèche, and J. M. Luck[26]).

1. INTRODUCTION

The question of determining the distribution of the occupation time of simple stochastic processes has been well debated by probabilists in the past.[1–5] The simplest example is that of the binomial random walk, or, in its continuum version, of Brownian motion. Let x_t be the position of the walker at time t, where t is an integer, for the random walker, or a continuous variable, for Brownian motion. Then define the stochastic process $\sigma_t = \text{sign } x_t$. The occupation times T_t^+ and T_t^- are the lengths of time spent

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by the walker, respectively on the right side, or on the left side of the origin, up to time t (using continuous time):

$$T_t^{\pm} = \int_0^t dt' \frac{1 \pm \sigma_{t'}}{2}. \tag{1.1}$$

It is also convenient to define the more symmetrical quantity

$$S_t = tM_t = \int_0^t dt' \sigma_{t'} = T_t^+ - T_t^-. \tag{1.2}$$

M_t, the mean of the stochastic process σ_t, will be hereafter referred to as the mean magnetization, by analogy with physical situations where σ_t is the spin at a given point of space.

A classical result, due to P. Lévy, is that the limiting distribution of the fraction of time spent by the walker on one side of the origin up to time t, as $t \to \infty$, is given by the arcsine law[3]

$$\lim_{t \to \infty} f_{t^{-1}T^{\pm}}(x) = \frac{1}{\pi \sqrt{x(1-x)}} \qquad (0 < x < 1) \tag{1.3}$$

or equivalently, for the mean magnetization,

$$f_M(x) = \frac{1}{\pi \sqrt{1-x^2}} \qquad (-1 < x < 1). \tag{1.4}$$

The interpretation of this result is that, contrarily to intuition, the random walker spends most of its time on one side of the origin. Translated into the language of a game of chance, this amounts to saying that one of the players is almost always winning, or almost always loosing, a situation which can be summarized by the "persistence of luck, or of bad luck".

On the other hand, a well-known result is that the probability for the random walker to remain on one side of the origin up to time t, i.e., for the stochastic process σ_t not to change sign up to time t, decays as $t^{-\frac{1}{2}}$, which defines the first-passage exponent $\frac{1}{2}$. The identity between this exponent and that characterizing the singularity of the probability density functions (1.3, 1.4) for $x \to 1$ is not coincidental. Both exponents are actually intimately related, as will be explained later in more generality.

The stochastic process σ_t defined above is simple in two respects. Firstly, the steps of the walker, or the increments of Brownian motion, are independent. Secondly, the process is zero-dimensional, in the sense that it does not interact with other processes.

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[3] See appendix A for the notation.

The question of *phase persistence* for self-similar growing systems, which appeared more recently in statistical physics, is a natural generalization of the probabilistic problem presented above. The physical systems in consideration are for instance breath figures, the pattern formed by growing and coalescing water droplets on a plane;[6, 7] systems of spins quenched from high temperature to zero temperature, or more generally to the low-temperature phase;[8] or the diffusion field evolving in time from a random initial condition.[9] The question posed is: what fraction of space remained in the same phase up to time t, or equivalently, what is the probability for a given point of space to remain in this phase. In this context, the stochastic process σ_t is, for instance, the indicator of whether a given point of space is in the dry phase for the breath figure experiment, the spin at a given site of a lattice for the zero-temperature kinetic Ising model, or the sign of the field at a given point in space for the diffusion equation.

In these cases, the probability for a given point of space to remain in a given phase is equal to the probability $p_0(t)$ that the process σ_t did not change sign up to time t, or persistence probability. For these situations, and similar ones, $p_0(t)$ decays algebraically with an exponent θ, the persistence exponent, which plays the role in the present context of the first-passage exponent $\frac{1}{2}$ of the random walk.

Phase persistence is far more complex than the simple case of the persistence of luck for the random walker, because the physical systems considered are spatially extended. Thus, the stochastic process σ_t, which is defined at a given point of space, interacts with similar processes at other points of space. As a consequence, the persistence exponent θ is usually hard to determine analytically.

Pursuing the parallel with the random walk, one may wonder, for the physical systems where phase persistence is observed, what is the behavior of the distribution of the occupation time (1.1), or of the mean magnetization (1.2).

For the zero-temperature Glauber-Ising chain,[10] the diffusion equation,[10, 11] the two-dimensional Ising model,[12] or for growing surfaces,[13] non-trivial U-shaped distributions f_M are observed. The singularity exponent of these distributions, as $M_t \rightarrow \pm 1$, gives the persistence exponent, thus providing a stationary definition of persistence, even at finite temperature.[12] The only analytical results at our disposal for these examples is for the diffusion equation, using the so-called independent-interval approximation.[10] However, understanding the origin of the existence of a limiting distribution is easy. In the long-time regime the two-time auto-correlation is a function of the ratio of the two times. Therefore the variance of M_t remains finite (see equation (9.3) below and ref. 10).

Another case of interest is the voter model. The investigation of the distribution of the occupation time for this model was introduced by Cox and Griffeath[14] (see ref. 15 for a summary). In contrast with the former examples, persistence is not algebraic for this case,[16, 17] and the distribution f_M is slowly peaking with time ($\langle M_t^2 \rangle \sim 1/\ln t$). This is akin to what occurs in the Ising model quenched from high temperature to the critical point. The explanation is the same in both cases: the two-time autocorrelation is the product of a scaling function of the ratio of the two times by a prefactor depending on one of the two times. In the case of the Ising model, this prefactor is related to the anomalous dimension of the field at criticality (see equation (10.7) below and ref. 18). As a consequence, the variance of M_t is slowly decreasing in time ($\langle M_t^2 \rangle \sim t^{-2\beta/\nu z_c}$). Let us recall that the voter model is critical.[19] The slow peaking of f_M for both models is the signature of criticality.

To summarize at this point, there is a strong analogy between the persistence of luck for the random walk, and phase persistence for self-similar coarsening systems. However, for the latter, both the first-passage (or persistence) exponent θ and the limiting distributions $\lim_{t \to \infty} f_{t^{-1}T^\pm}(x)$ and $f_M(x)$ are difficult to determine analytically.

Therefore, instead of considering spatially extended coarsening systems, we adopt the strategy of investigating simpler (zero-dimensional) stochastic processes, where the persistence exponent is a parameter in the definition of the model, yet where the distribution of the occupation time, or of the magnetization, is non-trivial. So doing we shall gain a better understanding of the nature of the persistent events encoded in these distributions, and shall be better prepared to investigate the statistics of the occupation time for the more difficult statistical mechanical models.

The present work is devoted to the study of a first example of this category of problems, where the stochastic process σ_t is generated by a renewal process, defined as follows. Events occur at the random epochs of time $t_1, t_2, \ldots,$ from some time origin $t = 0$. These events are considered as the zero crossings of the stochastic process $\sigma_t = \pm 1$. We take the origin of time on a zero crossing. This process is known as a *point process*. When the intervals of time between events, $\tau_1 = t_1, \tau_2 = t_2 - t_1, \ldots,$ are independent and identically distributed random variables with density $\rho(\tau)$, the process thus formed is a *renewal process*. Hereafter we shall use indifferently the denominations: events, zero crossings or renewals.

In this model the persistence probability $p_0(t)$, that is the probability that no event occurred up to time t, is simply given by the tail probability:

$$p_0(t) = \mathcal{P}(\tau > t) = \int_t^\infty d\tau \, \rho(\tau). \tag{1.5}$$

The approach we use is systematic and applies to any distribution ρ. In what follows ρ will be either a narrow distribution with finite moments, in which case the decay of $p_0(t)$, as $t \to \infty$, is faster than any power law, or a broad distribution characterized by a power-law fall-off with index θ:

$$\int_t^\infty d\tau \, \rho(\tau) \approx \left(\frac{\tau_0}{t}\right)^\theta \qquad (0 < \theta < 2), \tag{1.6}$$

where τ_0 is a microscopic time scale. If $\theta < 1$ all moments of ρ are divergent, while if $1 < \theta < 2$, the first moment $\langle \tau \rangle$ is finite but higher moments are divergent. In Laplace space, where s is conjugate to τ, for a narrow distribution we have

$$\mathop{\mathscr{L}}_{\tau} \rho(\tau) = \hat{\rho}(s) \mathop{=}_{s \to 0} 1 - \langle \tau \rangle \, s + \tfrac{1}{2} \langle \tau^2 \rangle \, s^2 + \cdots \tag{1.7}$$

For a broad distribution, (1.6) yields

$$\hat{\rho}(s) \mathop{\approx}_{s \to 0} \begin{cases} 1 - a \, s^\theta & (\theta < 1) \\ 1 - \langle \tau \rangle \, s + a \, s^\theta & (1 < \theta < 2), \end{cases} \tag{1.8}$$

with $a = |\Gamma(1 - \theta)| \, \tau_0^\theta$.

The case $\theta = \frac{1}{2}$ accounts for Brownian motion. Indeed, as is well known, the distribution of first-passage times by the origin behaves at large times as $\tau^{-\frac{3}{2}}$, i.e., is in the basin of attraction of a Lévy law of index $\frac{1}{2}$.[20] This can be simply worked out in the case of the random walk, where the discreteness of time allows a natural regularization of the process at short times. Hence, since we are interested in universal asymptotic properties of the distribution of the mean magnetization, the description of Brownian motion by a renewal process with distribution of intervals given by (1.6) or (1.8), with index $\theta = \frac{1}{2}$, is faithful.

Finally, let us give the content of the present work and stress its originality.

Though the study of renewal processes is classical,[20–22] fewer references are devoted to renewal processes with broad distributions of intervals. Elements on this question can be found in refs. 20 and 23. The present work provides a systematic account of the theory. In particular we perform the scaling analysis of the distributions of the various random variables naturally occurring in a renewal process, such as the number of events between 0 and t, the epoch of the last event before t, the backward and forward times. We also analyze the aging (i.e., non-stationary) properties of the distribution of the number of events occurring between two arbitrary

instants of time and of the two-time correlation function, for the case of a broad distribution ρ with $\theta < 1$.

The study of the limiting distribution of the occupation time of renewal processes is the subject of the work of Lamperti.[5] His main result is the expression of this distribution for a broad law of intervals ρ, with $\theta < 1$ (see equation (7.10) below). This result is recovered by a simple method in ref. 24, for the case where ρ is a stable Lévy distribution of intervals, with $\theta < 1$. However neither the work of Lamperti, nor ref. 24, contain the analysis of the scaling of this quantity for the case $1 < \theta < 2$. The present work fills this gap.

Finally, the methods used in the present work provide a firm basis to the investigation of the distribution of the occupation time for the process with time-dependent noise considered in ref. 25. This model is a second example of the category of problems mentioned above. There again, the existence of a renewal process in the model plays a crucial role. This will be the subject of a joint paper.[26] Both examples, the renewal processes considered in the present work, and that just mentioned, are deformations of the binomial random walk, or in continuous time, of Brownian motion. However they lead to non-trivial limiting distributions of the occupation time and of related quantities, as $t \to \infty$.

2. OBSERVABLES OF INTEREST

Let us introduce the quantities, the distributions of which will be computed in the following sections.

First, the number of events which occurred between 0 and t, denoted by N_t, is the random variable for the largest n for which $t_n \leqslant t$. The time of occurrence of the last event before t, that is of the N_t-th event, is therefore[4]

$$t_N = \tau_1 + \cdots + \tau_N.$$

The backward recurrence time B_t is defined as the length of time measured backwards from t to the last event before t, i.e.,

$$B_t = t - t_N,$$

while the forward recurrence time (or excess time) E_t is the time interval between t and the next event,

$$E_t = t_{N+1} - t.$$

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[4] We drop the time dependence of the random variable when it is in subscript.

The occupation times T_t^+ and T_t^-, i.e., the lengths of time spent by the σ-process, respectively in the $+$ and $-$ states, up to time t, were defined in the introduction as

$$T_t^\pm = \int_0^t dt' \frac{1 \pm \sigma_{t'}}{2},$$

hence $t = T_t^+ + T_t^-$. They are simply related to the sum S_t by

$$S_t = \int_0^t dt' \sigma_{t'} = T_t^+ - T_t^- = 2T_t^+ - t = t - 2T_t^-.$$

Assume that $\sigma_{t=0} = +1$. Then

$$\left.\begin{array}{l} T_t^+ = \tau_1 + \tau_3 + \cdots + \tau_N \\ T_t^- = \tau_2 + \tau_4 + \cdots + \tau_{N-1} + B_t \end{array}\right\} \text{if } N_t = 2k+1 \text{ (i.e., } \sigma_t = -1) \qquad (2.1)$$

and

$$\left.\begin{array}{l} T_t^+ = \tau_1 + \tau_3 + \cdots + \tau_{N-1} + B_t \\ T_t^- = \tau_2 + \tau_4 + \cdots + \tau_N \end{array}\right\} \text{if } N_t = 2k \text{ (i.e., } \sigma_t = +1). \qquad (2.2)$$

Assume now that $\sigma_{t=0} = -1$. Then, with obvious notations, the following relation holds

$$T_t^\pm(\sigma_{t=0} = -1) = T_t^\mp(\sigma_{t=0} = +1), \qquad (2.3)$$

hence

$$S_t(\sigma_{t=0} = -1) = -S_t(\sigma_{t=0} = +1). \qquad (2.4)$$

Finally we shall also be interested in two-time quantities, namely the number of zero crossings which occurred between t and $t+t'$, given by $N(t, t+t') = N_{t+t'} - N_t$, and the two-time autocorrelation of the process σ_t, defined as $C(t, t+t') = \langle \sigma_t \sigma_{t+t'} \rangle$.

3. NUMBER OF RENEWALS BETWEEN 0 AND t

The probability distribution of the number of events N_t between 0 and t reads

$$p_n(t) = \mathscr{P}(N_t = n) = \mathscr{P}(t_n < t < t_{n+1}) = \langle I(t_n < t < t_{n+1}) \rangle \qquad (n \geqslant 0),$$
$$(3.1)$$

where $I(t_n < t < t_{n+1}) = 1$ if the event inside the parenthesis occurs, and 0 if not. Note that $t_0 = 0$. The brackets denotes the average over $\tau_1, \tau_2, \ldots$ The case $n = 0$ is accounted for by equation (1.5).

Laplace transforming equation (3.1) with respect to t yields

$$\mathcal{L}_t p_n(t) = \hat{p}_n(s) = \left\langle \int_{t_n}^{t_{n+1}} dt\, \mathrm{e}^{-st} \right\rangle = \left\langle \mathrm{e}^{-st_n} \frac{1 - \mathrm{e}^{-s\tau_{n+1}}}{s} \right\rangle \qquad (n \geqslant 0), \quad (3.2)$$

and therefore

$$\hat{p}_n(s) = \hat{\rho}(s)^n \frac{1 - \hat{\rho}(s)}{s} \qquad (n \geqslant 0). \tag{3.3}$$

This distribution is normalized since $\sum_{n=0}^{\infty} \hat{p}_n(s) = 1/s$.

From (3.3) one can easily obtain the moments of N_t in Laplace space. For instance

$$\mathcal{L}_t \langle N_t \rangle = \sum_{n=1}^{\infty} n \hat{p}_n(s) = \frac{\hat{\rho}(s)}{s(1 - \hat{\rho}(s))}, \tag{3.4}$$

$$\mathcal{L}_t \langle N_t^2 \rangle = \sum_{n=1}^{\infty} n^2 \hat{p}_n(s) = \frac{\hat{\rho}(s)(1 + \hat{\rho}(s))}{s(1 - \hat{\rho}(s))^2}. \tag{3.5}$$

We now discuss the above results according to the nature of the distribution of intervals $\rho(\tau)$.

(i) *Narrow distributions of intervals*

Expanding (3.4) and (3.5) as series in s, and performing the inverse Laplace transform term by term, yields

$$\langle N_t \rangle \underset{t \to \infty}{\approx} \frac{t}{\langle \tau \rangle} + c_1,$$

$$\langle N_t^2 \rangle - \langle N_t \rangle^2 \underset{t \to \infty}{\approx} \frac{\langle \tau^2 \rangle - \langle \tau \rangle^2}{\langle \tau \rangle^3} t + c_2,$$

where the constants c_1 and c_2 can be expressed in terms of the moments of ρ. More generally, all the cumulants of N_t scale as t, as would be the case for the sum of t independent random variables, and therefore N_t obeys the central limit theorem.[21, 22]

(ii) *Broad distributions of intervals with index $\theta < 1$*
Using equation (1.8), (3.4) yields

$$\langle N_t \rangle \underset{t \to \infty}{\approx} \frac{\sin \pi\theta}{\pi\theta} \left(\frac{t}{\tau_0} \right)^\theta, \tag{3.6}$$

while (3.5) yields $\langle N_t^2 \rangle \sim t^{2\theta}$. More generally one expects that the cumulant of order k of N_t scales as $t^{k\theta}$. Indeed, setting

$$N_t = \frac{(t/\tau_0)^\theta}{\Gamma(1-\theta)} X_t,$$

where we keep the same notation X_t for the scaling variable, we obtain, using (1.8) and

$$p_n(t) = \int \frac{ds}{2\pi i} e^{st} \hat{\rho}(s)^n \frac{1 - \hat{\rho}(s)}{s}, \tag{3.7}$$

the limiting distribution of X_t, as $t \to \infty$,

$$f_X(x) = \int \frac{dz}{2\pi i} z^{\theta-1} e^{z - xz^\theta} \qquad (0 < x < \infty). \tag{3.8}$$

For small x, expanding the integrand in the right side and folding the contour around the negative real axis yields

$$f_X(x) \underset{x \to 0}{=} \frac{1}{\Gamma(1-\theta)} - \frac{x}{\Gamma(1-2\theta)} + \cdots$$

At large x, applying the steepest-descent method, we find the stretched exponential fall-off

$$f_X(x) \underset{x \to \infty}{\sim} \exp(-(1-\theta)(\theta^\theta x)^{1/(1-\theta)}),$$

which demonstrates that all the moments of X are finite.
For $\theta = 1/2$, the distribution of X is given by a half Gaussian:

$$f_X(x) = \frac{e^{-x^2/4}}{\sqrt{\pi}} \qquad (0 < x < \infty).$$

(iii)   *Broad distributions of intervals with index* $1 < \theta < 2$

Using equation (1.8), we obtain

$$\langle N_t \rangle \underset{t \to \infty}{\approx} \frac{t}{\langle \tau \rangle} + \frac{\tau_0^\theta}{(\theta - 1)(2 - \theta)\langle \tau \rangle^2} t^{2 - \theta}, \tag{3.9}$$

$$\langle N_t^2 \rangle - \langle N_t \rangle^2 \underset{t \to \infty}{\approx} \frac{2\tau_0^\theta}{(2 - \theta)(3 - \theta)\langle \tau \rangle^3} t^{3 - \theta}. \tag{3.10}$$

The fluctuations of the variable N_t around its mean $t/\langle \tau \rangle$ are therefore no longer characterized by a single scale. These fluctuations are very large as we now show.

We set, still keeping the same notation X_t for the scaling variable,

$$N_t = \frac{t}{\langle \tau \rangle} + \frac{\tau_0}{\langle \tau \rangle} \left(-\Gamma(1 - \theta) \frac{t}{\langle \tau \rangle} \right)^{1/\theta} X_t. \tag{3.11}$$

Then, using equations (1.8) and (3.7), we obtain the limiting distribution of X_t, as $t \to \infty$,

$$f_X(x) = \int \frac{dz}{2\pi i} \, e^{-zx + z^\theta}. \tag{3.12}$$

For large positive values of x, f_X falls off exponentially as

$$f_X(x) \underset{x \to +\infty}{\sim} \exp\left(-(\theta - 1)\left(\frac{x}{\theta} \right)^{\theta/(\theta - 1)} \right).$$

For large negative values of x, linearizing the integrand of equation (3.12) with respect to z^θ, and folding the contour around the negative real axis, we find

$$f_X(x) \underset{x \to -\infty}{\approx} \frac{|x|^{-\theta - 1}}{\Gamma(-\theta)}.$$

As a consequence, $\langle X_t \rangle$ is finite, and all higher order moments diverge. Actually $\langle X_t \rangle$ vanishes, as seen from (3.9) and (3.11), because the difference of exponents $2 - \theta - 1/\theta = -(\theta - 1)^2/\theta$ is negative. One can also check on (3.10) that $\langle X_t^2 \rangle$ is divergent, since the difference of exponents $3 - \theta - 2/\theta = (\theta - 1)(2 - \theta)/\theta$ is now positive.

In the limit $\theta \to 2$, f_X is a Gaussian:

$$f_X(x) \underset{\theta \to 2}{\to} \frac{e^{-x^2/4}}{2\sqrt{\pi}}.$$

Related considerations can be found in ref. 20, vol. 2, p. 373.

4. EPOCH OF LAST RENEWAL *tN*

We begin our study of the distributions of the random variables t_N, B_t, E_t, $T_t^{\pm}$, and S_t appearing in section 2 by the example of t_N, the epoch of the last renewal before t.

This quantity is the sum of a random number N_t of random variables $\tau_1, \tau_2, ...$, i.e., it is a function of the joint random variables $\{\tau_1, \tau_2, ..., \tau_N\}$. Now, the latter are *not* independent, although the intervals $\tau_1, \tau_2, ...$ are, by definition, independent. This is due to the very definition of N_t. Indeed if, say, N_t takes the value n, then the sum $t_n = \sum_{i=1}^{n} \tau_i$ is constrained to be less than t. Therefore, in particular, each individual interval τ_i is constrained to be less than t.

These considerations will be now made more precise, by the computation of the distribution of t_N, below, and by that of the joint probability density function of $\{\tau_1, \tau_2, ..., \tau_N\}$ and N_t, in the next section.

The joint probability distribution $f_{t_N, N}$ of the random variables t_N and N_t reads

$$f_{t_N, N}(t; y, n) = \frac{d}{dy} \mathscr{P}(t_N < y, N_t = n) = \langle \delta(y - t_N) \, I(t_n < t < t_{n+1}) \rangle,$$

from which one deduces the density f_{t_N} of t_N

$$f_{t_N}(t; y) = \frac{d}{dy} \mathscr{P}(t_N < y) = \langle \delta(y - t_N) \rangle = \sum_{n=0}^{\infty} f_{t_N, N}(t; y, n).$$

In Laplace space, where s is conjugate to t and u to y,

$$\mathop{\mathscr{L}}_{t, y} f_{t_N, N}(t; y, n) = \hat{f}_{t_N, N}(s; u, n) = \left\langle e^{-ut_n} \int_{t_n}^{t_{n+1}} dt \, e^{-st} \right\rangle$$

$$= \hat{\rho}(s+u)^n \frac{1 - \hat{\rho}(s)}{s} \qquad (n \geqslant 0). \tag{4.1}$$

The distribution (3.3) of N_t is recovered by setting $u = 0$ in (4.1). Summing over n gives the distribution of t_N, in Laplace space:

$$\mathop{\mathscr{L}}_{t} \langle e^{-ut_N} \rangle = \hat{f}_{t_N}(s; u) = \frac{1}{1 - \hat{\rho}(s+u)} \frac{1 - \hat{\rho}(s)}{s}, \tag{4.2}$$

which is normalized since $\hat{f}_{t_N}(s; u = 0) = 1/s$.

The case where the distribution of intervals $\rho(\tau)$ is broad, with $\theta < 1$ (see equation (1.8)), is of particular interest since it leads to a limiting distribution for the random variable t_N/t.

In the long-time scaling regime, where t and t_N are both large and comparable, or u, s small and comparable, we get

$$\hat{f}_{t_N}(s; u) \approx s^{\theta-1}(s+u)^{-\theta}.$$

This yields, using the method of appendix B, the limiting distribution for the rescaled variable $t^{-1}t_N$, as $t \to \infty$,

$$\lim_{t \to \infty} f_{t^{-1}t_N}(x) = \frac{\sin \pi\theta}{\pi} x^{\theta-1}(1-x)^{-\theta} = \beta_{\theta, 1-\theta}(x) \qquad (0 < x < 1), \qquad (4.3)$$

with $x = y/t$, and where

$$\beta_{a, b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} x^{a-1}(1-x)^{b-1}$$

is the beta distribution on $[0, 1]$. As a consequence

$$\langle t_N \rangle \underset{t \to \infty}{\approx} \theta t. \qquad (4.4)$$

In the particular case $\theta = \frac{1}{2}$, we have

$$\lim_{t \to \infty} f_{t^{-1}t_N}(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad (4.5)$$

which is the arcsine law on $[0, 1]$. This is a well-known property of Brownian motion.[20]

5. INTERDEPENDENCE OF $\{\tau_1, ..., \tau_N\}$

The purpose of this section is to identify the interdependence of the N_t first time intervals $\{\tau_1, \tau_2, ..., \tau_N\}$, a point both of conceptual and of practical importance (see e.g. ref. 26).

In order to do so, we compute the joint probability distribution of these random variables and of N_t, denoted by

$$f_{\{\tau_1, \tau_2, ..., \tau_N\}, N}(t; \tau_1, \tau_2, ..., \tau_n, n).$$

Generalizing the calculations done above, one has, in the Laplace space of all temporal variables,

$$\hat{f}_{\{\tau_1, \tau_2, \ldots, \tau_N\}, N}(s; u_1, \ldots, u_n, n) = \left\langle \prod_{i=1}^{n} e^{-u_i \tau_i} \int_{t_n}^{t_{n+1}} dt \, e^{-st} \right\rangle,$$

resulting in

$$\hat{f}_{\{\tau_1, \tau_2, \ldots, \tau_N\}, N}(s; u_1, u_2, \ldots, u_n, n) = \frac{1 - \hat{\rho}(s)}{s} \prod_{i=1}^{n} \hat{\rho}(s + u_i) \qquad (n \geqslant 0),$$

where the empty product is equal to 1 for $n = 0$.

The marginal distribution (3.3) of N_t can be recovered by setting all the $u_i = 0$ in the above expression. By inversion with respect to the variables $\{u_i\}$, one gets

$$\hat{f}_{\{\tau_1, \tau_2, \ldots, \tau_N\}, N}(s; \tau_1, \tau_2, \ldots, \tau_n, n) = \frac{1 - \hat{\rho}(s)}{s} \prod_{i=1}^{n} \rho(\tau_i) \, e^{-s\tau_i} \qquad (5.1)$$

and finally

$$f_{\{\tau_1, \tau_2, \ldots, \tau_N\}, N}(t; \tau_1, \tau_2, \ldots, \tau_n, n) = \left(\prod_{i=1}^{n} \rho(\tau_i) \right) \mathscr{P}(N_{t-t_n} = 0) \, \Theta(t - t_n),$$

where $\Theta(x)$ is Heaviside step function. This expression clearly exhibits the interdependence of the random variables $\{\tau_1, \tau_2, \ldots, \tau_N\}$.

Let us investigate the distribution of any one of the τ_i, say τ_1. We denote this random variable by τ_t in order to enhance the fact that its distribution is constrained. By integration of (5.1) on $\tau_2, \tau_3, \ldots, \tau_n$, we obtain the joint distribution of N_t and τ_t in Laplace space:

$$\hat{f}_{\tau_t, N}(s; \tau, n) = \frac{1 - \hat{\rho}(s)}{s} \times \begin{cases} \hat{\rho}(s)^{n-1} \, e^{-s\tau} \rho(\tau) & (n \geqslant 1), \\ \delta(\tau) & (n = 0), \end{cases}$$

from which one deduces, by summation upon n, that

$$\hat{f}_{\tau_t}(s; \tau) = \rho(\tau) \frac{e^{-s\tau}}{s} + \delta(\tau) \frac{1 - \hat{\rho}(s)}{s}. \qquad (5.2)$$

This can be finally inverted, yielding

$$f_{\tau_t}(t; \tau) = \rho(\tau) \, \Theta(t - \tau) + \delta(\tau) \, p_0(t). \qquad (5.3)$$

This result can be interpreted as follows. The a priori distribution $\rho(\tau)$ is unaffected as long as τ is less than the observation time t. The complementary event, which has probability $p_0(t)$, corresponds formally to $\tau = 0$.

As a consequence of (5.2) and (5.3) we have

$$\mathscr{L}_t \langle \tau_t \rangle = \frac{1}{s} \int_0^\infty d\tau \, \tau \, \rho(\tau) \, e^{-s\tau} = -\frac{1}{s} \frac{d\hat{\rho}(s)}{ds}, \qquad (5.4)$$

and

$$\langle \tau_t \rangle = \int_0^\infty d\tau \, \tau \, f_{\tau_t}(t; \tau) = \int_0^t d\tau \, \tau \, \rho(\tau) = \langle \tau \rangle - \int_t^\infty d\tau \, \tau \, \rho(\tau), \qquad (5.5)$$

where the last expression holds when the a priori average $\langle \tau \rangle$ is finite.

Let us discuss the above results according to the nature of the distribution of intervals $\rho(\tau)$.

(i)   *Narrow distributions of intervals*

If the distribution ρ is narrow, then (5.5) shows that $\langle \tau_t \rangle$ converges to $\langle \tau \rangle$ very rapidly. For instance, if ρ is exponential, then $\langle \tau_t \rangle = [1 - (1 + \lambda t) e^{-\lambda t}]/\lambda$: the decay of $\langle \tau \rangle - \langle \tau_t \rangle$, with $\langle \tau \rangle = 1/\lambda$, is exponential.

(ii)   *Broad distributions of intervals with index $\theta < 1$*
If the distribution ρ is broad, with $\theta < 1$, then from (5.4)

$$\langle \tau_t \rangle \underset{t \to \infty}{\approx} \frac{\theta \tau_0^\theta}{1 - \theta} t^{1-\theta}.$$

The interpretation of this last result is that $t_N \sim t$ is the sum of $N_t \sim t^\theta$ time intervals $\tau_t \sim t^{1-\theta}$. However $\langle t_N \rangle \equiv \langle N_t \tau_t \rangle \neq \langle N_t \rangle \langle \tau_t \rangle$ (see equations (3.6) and (4.4)).

(iii)   *Broad distributions of intervals with index $1 < \theta < 2$*
We now obtain

$$\langle \tau_t \rangle \underset{t \to \infty}{\approx} \langle \tau \rangle - \frac{\theta \tau_0^\theta}{\theta - 1} t^{-(\theta-1)},$$

showing that $\langle \tau_t \rangle$ converges to $\langle \tau \rangle$ very slowly.

6. BACKWARD AND FORWARD RECURRENCE TIMES

The distributions of $B_t = t - t_N$ and $E_t = t_{N+1} - t$ can be obtained using the methods of the previous sections. We have

$$f_{B,N}(t; y, n) = \langle \delta(y - t + t_n) \, I(t_n < t < t_{n+1}) \rangle,$$
$$f_{E,N}(t; y, n) = \langle \delta(y - t_{n+1} + t) \, I(t_n < t < t_{n+1}) \rangle.$$

In Laplace space, where s is conjugate to t and u to y,

$$\mathop{\mathscr{L}}_{t,y} f_{B,N}(t; y, n) = \hat{f}_{B,N}(s; u, n) = \left\langle \int_{t_n}^{t_{n+1}} dt \, e^{-st} e^{-u(t - t_n)} \right\rangle,$$
$$\mathop{\mathscr{L}}_{t,y} f_{E,N}(t; y, n) = \hat{f}_{E,N}(s; u, n) = \left\langle \int_{t_n}^{t_{n+1}} dt \, e^{-st} e^{-u(t_{n+1} - t)} \right\rangle,$$

hence, for $n \geqslant 0$,

$$\hat{f}_{B,N}(s; u, n) = \hat{\rho}(s)^n \frac{1 - \hat{\rho}(s + u)}{s + u},$$
$$\hat{f}_{E,N}(s; u, n) = \hat{\rho}(s)^n \frac{\hat{\rho}(s) - \hat{\rho}(u)}{u - s},$$

and therefore

$$\hat{f}_B(s; u) = \frac{1 - \hat{\rho}(s + u)}{s + u} \frac{1}{1 - \hat{\rho}(s)}, \tag{6.1}$$

$$\hat{f}_E(s; u) = \frac{\hat{\rho}(u) - \hat{\rho}(s)}{s - u} \frac{1}{1 - \hat{\rho}(s)}. \tag{6.2}$$

Equations (6.1) and (4.2) are related by $\hat{f}_B(s; u) = \hat{f}_{t_N}(s + u; -u)$, expressing the identity $t_N + B_t = t$.

We now discuss the above results according to the nature of the distribution $\rho(\tau)$.

(i) Narrow distributions of intervals

For distributions with finite moments, equilibrium is attained at long times, for both the backward and forward recurrence times, with a common distribution given in Laplace space by

$$\hat{f}_{B,\text{eq}}(u) = \hat{f}_{E,\text{eq}}(u) = \lim_{s \to 0} s \hat{f}_B(s; u) = \lim_{s \to 0} s \hat{f}_E(s; u) = \frac{1 - \hat{\rho}(u)}{\langle \tau \rangle u}. \tag{6.3}$$

By inversion we obtain

$$f_{B,\,\text{eq}}(y) = f_{E,\,\text{eq}}(y) = \frac{1}{\langle \tau \rangle} \int_y^\infty d\tau\, \rho(\tau) = \frac{p_0(y)}{\langle \tau \rangle}. \qquad (6.4)$$

In the particular case where ρ is exponential, inversion of (6.1) yields, for finite t,

$$f_B(t;\, y) = \lambda e^{-\lambda y} \Theta(t-y) + e^{-\lambda t} \delta(t-y).$$

The weight of the second term is simply $\mathscr{P}(B_t = t) = p_0(t)$. Similarly, by inversion of (6.2), we have

$$f_E(t;\, y) = f_{E,\,\text{eq}}(y) = f_{B,\,\text{eq}}(y) = \lambda e^{-\lambda y} = \rho(y).$$

(ii)   *Broad distributions of intervals with index $\theta < 1$*

We obtain, in the long-time scaling regime, i.e., for u, s small and comparable,

$$\hat{f}_B(s;\, u) \approx s^{-\theta}(s+u)^{\theta-1},$$

which, by inversion, using the method of appendix B, yields the limiting distribution

$$\lim_{t \to \infty} f_{t^{-1}B}(x) = \frac{\sin \pi \theta}{\pi} x^{-\theta}(1-x)^{\theta-1} = \beta_{1-\theta,\,\theta}(x) \qquad (0 < x < 1). \qquad (6.5)$$

This result is consistent with (4.3). Similarly,

$$\hat{f}_E(s;\, u) \approx \frac{u^\theta - s^\theta}{s^\theta(u-s)},$$

yielding

$$\lim_{t \to \infty} f_{t^{-1}E}(x) = \frac{\sin \pi \theta}{\pi} \frac{1}{x^\theta(1+x)} \qquad (0 < x < \infty). \qquad (6.6)$$

Let us point out that the limiting distribution of t/t_{N+1}, as $t \to \infty$, is given by (4.3).

For $\theta = \frac{1}{2}$, we obtain

$$\lim_{t \to \infty} f_{t^{-1}B}(x) = \frac{1}{\pi \sqrt{x(1-x)}} \qquad (0 < x < 1),$$

which is the arcsine law on $[0, 1]$. Similarly

$$\lim_{t \to \infty} f_{t^{-1}E}(x) = \frac{1}{\pi(1+x)\sqrt{x}} \qquad (0 < x < \infty).$$

(iii) *Broad distributions of intervals with index $1 < \theta < 2$*

The backward and forward recurrence times still admit the common limiting distribution (6.4), which in the present case has the following asymptotic behavior

$$f_{B, \text{eq}}(y) = f_{E, \text{eq}}(y) \underset{y \to \infty}{\approx} \frac{1}{\langle \tau \rangle} \left(\frac{\tau_0}{y} \right)^{\theta}, \tag{6.7}$$

characteristic of a broad distribution of index $\theta - 1 < 1$. In other words, at equilibrium the average backward and forward recurrence times diverge. However their long-time behavior can be computed from (6.1) and (6.2), yielding

$$\mathscr{L}_t \langle B_t \rangle = \frac{1 - \hat{\rho}(s) + s \, d\hat{\rho}(s)/ds}{s^2(1 - \hat{\rho}(s))}, \qquad \mathscr{L}_t \langle E_t \rangle = \frac{\hat{\rho}(s) - 1 + \langle \tau \rangle s}{s^2(1 - \hat{\rho}(s))},$$

from which it follows that

$$\langle B_t \rangle \underset{t \to \infty}{\approx} \frac{\tau_0^{\theta}}{(2-\theta)\langle \tau \rangle} t^{2-\theta}, \qquad \langle E_t \rangle \underset{t \to \infty}{\approx} \frac{\tau_0^{\theta}}{(\theta-1)(2-\theta)\langle \tau \rangle} t^{2-\theta}.$$

7. OCCUPATION TIME AND MEAN MAGNETIZATION

The central investigation of the present work concerns the determination of the distributions of the occupation times $T_t^{\pm}$ and of the sum S_t (or of the mean magnetization M_t).

Let us denote by $f_{T_t^{\pm}}^{\sigma_0}$ and $f_S^{\sigma_0}$ the probability density functions of these quantities for a fixed value of $\sigma_0 \equiv \sigma_{t=0}$, and by $f_{T^{\pm}}$ and f_S the corresponding probability density functions after averaging over $\sigma_0 = \pm 1$ with equal weights:

$$f = \tfrac{1}{2}(f^+ + f^-).$$

The symmetry properties (2.3) and (2.4) imply that

$$f_{T^+}(t; y) = f_{T^-}(t; y), \qquad f_S(t; y) = f_S(t; -y). \tag{7.1}$$

Following the methods used in the previous sections, we define the joint probability distribution of T_t^+ and N_t at fixed σ_0 as

$$f_{T^+, N}^{\sigma_0}(t; y, n) = \frac{d}{dy} \mathscr{P}(T_t^+ < y, N_t = n) = \langle \delta(y - T_t^+) \, I(t_n \leqslant t < t_{n+1}) \rangle,$$

and

$$f_{T^+}^{\sigma_0}(t; y) = \sum_{n=0}^{\infty} f_{T^+, N}^{\sigma_0}(t; y, n) = \langle \delta(y - T_t^+) \rangle.$$

For $\sigma_0 = +1$, Laplace transforming with respect to t and y, using equations (2.1) and (2.2), we find

$$\hat{f}_{T^+, N}^+(s; u, 2k+1) = \hat{\rho}^{k+1}(s+u) \, \hat{\rho}^k(s) \frac{1 - \hat{\rho}(s)}{s},$$

$$\hat{f}_{T^+, N}^+(s; u, 2k) = \hat{\rho}^k(s+u) \, \hat{\rho}^k(s) \frac{1 - \hat{\rho}(s+u)}{s+u},$$

hence, summing over k,

$$\hat{f}_{T^+}^+(s; u) = \left(\frac{1 - \hat{\rho}(s+u)}{s+u} + \hat{\rho}(s+u) \frac{1 - \hat{\rho}(s)}{s} \right) \frac{1}{1 - \hat{\rho}(s) \, \hat{\rho}(s+u)}.$$

Using the property $T_t^- = t - T_t^+$, and (7.1), we get

$$\hat{f}_{T^+}(s; u) = \tfrac{1}{2} (\hat{f}_{T^+}^+(s; u) + \hat{f}_{T^+}^+(s+u; -u)) = \hat{f}_{T^+}(s+u; -u).$$

Similarly, using the property $S_t = 2T_t^+ - t$, and (7.1), we get

$$\hat{f}_S(s; u) = \hat{f}_S(s; -u) = \hat{f}_{T^+}(s-u; 2u).$$

(Here $\hat{f}_S(t; u)$ is the bilateral Laplace transform with respect to y. See appendix A for the notation.)

The final results read

$$\hat{f}_{T^\pm}(s; u) = \frac{2s(1 - \hat{\rho}(s+u) \, \hat{\rho}(s)) + u(1 + \hat{\rho}(s+u))(1 - \hat{\rho}(s))}{2s(s+u)(1 - \hat{\rho}(s+u) \, \hat{\rho}(s))},$$

$$\hat{f}_S(s; u) = \frac{s(1 - \hat{\rho}(s+u) \, \hat{\rho}(s-u)) + u(\hat{\rho}(s+u) - \hat{\rho}(s-u))}{(s^2 - u^2)(1 - \hat{\rho}(s+u) \, \hat{\rho}(s-u))}. \tag{7.2}$$

Let us discuss these results according to the nature of the distribution of intervals $\rho(\tau)$.

(i) *Narrow distributions of intervals*

If the distribution ρ is narrow, implying that the correlations between the sign process σ_t at two instants of time are short-ranged (see section 9), it is intuitively clear that S_t should scale as the sum of t independent random variables and therefore obey the central limit theorem. In the context of the present work, we are especially interested in large deviations, i.e., rare (persistent) events where S_t deviates from its mean. In particular, the probability that S_t is *equal* to t is identical to $p_0(t)$, the persistence probability. These points are now made more precise.

First, expanding the right-hand side of (7.2) to second order in u, and performing the inverse Laplace transform, yields

$$\langle S_t^2 \rangle \underset{t \to \infty}{\approx} \frac{\langle \tau^2 \rangle - \langle \tau \rangle^2}{\langle \tau \rangle} t. \tag{7.3}$$

More generally, it can be checked that all the cumulants of S_t scale as t. Then,

$$\hat{f}_S(t; u) = \langle e^{-uS_t} \rangle = e^{K_S(t; u)} \underset{t \to \infty}{\sim} e^{t\Phi(u)}, \tag{7.4}$$

where $K_S(t; u)$ is the generating function of cumulants of S_t, and $\Phi(u)$ is determined below. By inversion of (7.4),

$$f_S(t; y) \underset{t \to \infty}{\sim} \int \frac{du}{2i\pi} e^{uy + t\Phi(u)}.$$

In the large-deviation regime, i.e., when t and y are simultaneously large and $x = y/t$ finite, the saddle-point method yields

$$f_M(t; x) \underset{t \to \infty}{\sim} \int \frac{du}{2i\pi} e^{t[ux + \Phi(u)]} \underset{t \to \infty}{\sim} e^{-t\Sigma(x)},$$

where

$$\Sigma(x) = -\min_u (ux + \Phi(u)) \qquad (-1 < x < 1)$$

is the large-deviation function (or entropy) for M_t. The functions $\Sigma(x)$ and $\Phi(u)$ are mutual Legendre transforms:

$$\Sigma(x) + \Phi(u) = -ux, \qquad u = -\frac{d\Sigma}{dx}, \qquad x = -\frac{d\Phi}{du}.$$

Equation (7.4) implies that $\hat{f}_S(s; u)$ is singular for $s = \Phi(u)$, and therefore, using (7.2), that

$$\hat{\rho}(\Phi(u) + u)\, \hat{\rho}(\Phi(u) - u) = 1, \qquad (7.5)$$

which determines implicitly $\Phi(u)$. For $u \to 0$, using $\hat{\rho}(u) \approx 1 - u\langle\tau\rangle + \frac{1}{2} u^2\langle\tau^2\rangle$, equation (7.5) gives

$$\Phi(u) \underset{u \to 0}{\approx} \frac{\langle\tau^2\rangle - \langle\tau\rangle^2}{2\langle\tau\rangle}\, u^2,$$

hence

$$\Sigma(x) \underset{x \to 0}{\approx} \frac{\langle\tau\rangle}{2(\langle\tau^2\rangle - \langle\tau\rangle^2)}\, x^2,$$

yielding a Gaussian distribution for S_t, thus recovering the central limit theorem for this quantity.

In the particular case of an exponential distribution $\rho(\tau)$, all results become explicit. We have

$$\hat{f}_S(s; u) = \frac{2\lambda + s}{s^2 + 2\lambda s - u^2},$$

and

$$\Phi(u) = \sqrt{\lambda^2 + u^2} - \lambda, \qquad \Sigma(x) = \lambda(1 - \sqrt{1 - x^2}).$$

(ii) *Broad distributions of intervals with index $\theta < 1$*

Using equation (1.8), we have, in the long-time scaling regime where u and s are small and comparable,

$$\hat{f}_S(s; u) \approx \frac{(s+u)^{\theta-1} + (s-u)^{\theta-1}}{(s+u)^\theta + (s-u)^\theta},$$

which, by inversion, using the method of appendix B, yields, as $t \to \infty$, the limiting distribution for the mean magnetization M_t,

$$f_M(x) = \frac{2 \sin \pi\theta}{\pi}\, \frac{(1-x^2)^{\theta-1}}{(1+x)^{2\theta} + (1-x)^{2\theta} + 2 \cos \pi\theta (1-x^2)^\theta}. \qquad (7.6)$$

For $x \to 0$, the expansion

$$f_M(x) = \frac{2 \tan(\pi\theta/2)}{\pi}\left(1 + \frac{\cos^2(\pi\theta/2) - \theta^2}{\cos^2(\pi\theta/2)}\, x^2 + \cdots\right)$$

shows that $x = 0$ is a minimum of $f_M(x)$ for $\theta < \theta_c$, while it is a maximum for $\theta > \theta_c$, with $\theta_c = \cos(\pi\theta_c/2)$, yielding $\theta_c = 0.594611$.[24]

For $x \to \pm 1$, $f_M(x)$ diverges as

$$f_M(x) \underset{x \to \pm 1}{\approx} \approx \frac{\sin \pi\theta}{\pi} 2^{-\theta}(1 \mp x)^{\theta-1}. \tag{7.7}$$

Comparing the amplitude of this power-law divergence to that of the symmetric beta distribution over $[-1, 1]$ of same index,

$$\beta(x) = \frac{\Gamma(\theta + \frac{1}{2})}{\Gamma(\theta)\sqrt{\pi}}(1 - x^2)^{\theta-1},$$

we have

$$\lim_{x \to \pm 1} \frac{f_M(x)}{\beta(x)} = B(\theta) = \frac{\Gamma(\theta)}{\Gamma(2\theta)\Gamma(1-\theta)}.$$

The amplitude ratio $B(\theta)$ decreases from $B(0) = 2$ to $B(1) = 0$; it is equal to 1 for $\theta = \frac{1}{2}$ (see equation (7.9)).

Equation (B5) yields the moments of f_M,

$$\langle M^2 \rangle = 1 - \theta, \quad \langle M^4 \rangle = 1 - \frac{\theta}{3}(4 - \theta^2), \quad \langle M^6 \rangle = 1 - \frac{\theta}{15}(23 - 10\theta^2 + 2\theta^4), \tag{7.8}$$

and so on.

In the particular case $\theta = \frac{1}{2}$, equation (7.6) simplifies to

$$f_M(x) = \frac{1}{\pi\sqrt{1 - x^2}} = \beta(x), \tag{7.9}$$

which is the arcsine law on $[-1, 1]$.

The corresponding distribution for the occupation time is

$$\lim_{t \to \infty} f_{t^{-1}T^{\pm}}(x) = \frac{\sin \pi\theta}{\pi} \frac{x^{\theta-1}(1-x)^{\theta-1}}{x^{2\theta} + (1-x)^{2\theta} + 2\cos\pi\theta\, x^{\theta}(1-x)^{\theta}}, \tag{7.10}$$

a result originally found by Lamperti.[5] If $\theta = \frac{1}{2}$, the arcsine law on $[0, 1]$ is recovered:

$$\lim_{t \to \infty} f_{t^{-1}T^{+}}(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

A last point is that, in the persistence region, i.e., for $M_t \to 1$ and $t \to \infty$, $f_M(t; x)$ has a scaling form, as we now show. Two limiting behaviors are already known:

(i)   for $M_t = 1$, $f_M(t; x) = \frac{1}{2} p_0(t) \, \delta(x - 1)$, with $p_0(t) \sim (t/\tau_0)^{-\theta}$;

(ii) for $t = \infty$, the limiting distribution $f_M(x)$ is given by equation (7.7).

In order to interpolate between these two behaviors, we assume the scaling form

$$f_M(t; x) \sim \left(\frac{t}{\tau_0}\right)^{1-\theta} h\left(z = (1-x)\,\frac{t}{\tau_0}\right),$$

in the persistence region defined above, with z fixed. The known limiting behaviors imply

$$h(z) \underset{z \to \infty}{\approx} 2^{-\theta}\, \frac{\sin \pi\theta}{\pi}\, z^{\theta-1}, \qquad h(z) \underset{z \to 0}{\approx} \frac{1}{2}\, \delta(z). \tag{7.11}$$

The scaling function $h(z)$ can be computed in Laplace space as follows. We have

$$\hat{f}_S(t; u) = \langle \mathrm{e}^{-uS_t}\rangle = \left(\frac{\tau_0}{t}\right)^{\theta} \mathrm{e}^{-ut} \int_0^\infty dz\, \mathrm{e}^{ut_0 z} h(z).$$

Laplace transforming with respect to t, yields, in the limit $s + u \to 0$, with $u = O(1)$, since z is finite,

$$\hat{f}_S(s; u) \approx a(s+u)^{\theta-1} \int_0^\infty dz\, \mathrm{e}^{ut_0 z} h(z).$$

By identification of this expression with the corresponding estimate obtained in the same regime from (7.2), we have, with $v = -2u$,

$$\frac{1 + \hat{\rho}(v)}{2(1 - \hat{\rho}(v))} = \int_0^\infty dz\, \mathrm{e}^{-\frac{1}{2} v\tau_0 z} h(z) = \hat{h}\left(\frac{v\tau_0}{2}\right).$$

As a consequence, the function $h(z)$ is non universal since it depends on the details of the function $\rho(\tau)$. Universality is restored only in the two limits considered above (see (7.11)).

(iii) *Broad distributions of intervals with index $1 < \theta < 2$*

Using equation (1.8), (7.2) yields

$$\hat{f}_S(s; u) \approx \frac{2\langle\tau\rangle - a((s+u)^{\theta-1} + (s-u)^{\theta-1})}{2\langle\tau\rangle s - a((s+u)^{\theta} + (s-u)^{\theta})}, \tag{7.12}$$

in the scaling regime where both Laplace variables s and u are simultaneously small and comparable. Expanding this expression as a Taylor series in u, and performing the inverse Laplace transform term by term, we obtain, for the first even moments of the random variable S_t,

$$\langle S_t^2 \rangle \underset{t\to\infty}{\approx} \frac{2\tau_0^{\theta}}{(2-\theta)(3-\theta)\langle\tau\rangle} t^{3-\theta},$$

$$\langle S_t^4 \rangle \underset{t\to\infty}{\approx} \frac{4\tau_0^{\theta}}{(4-\theta)(5-\theta)\langle\tau\rangle} t^{5-\theta}. \tag{7.13}$$

This demonstrates that, in the long-time regime, the asymptotic distribution of S_t is broad, with slowly decaying tails. This distribution can be evaluated as follows.

We first notice that equation (7.12) further simplifies for $|u| \gg s$. In order for both variables of $\hat{f}_S(s; u)$ to stay in the appropriate domains, we consider $s > 0$ and $u = \mathrm{i}\omega$, with ω real. We thus obtain, in the relevant regime ($s \ll |\omega| \ll 1$),

$$\hat{f}_S(s; u) \approx \frac{1}{s + c\,|\omega|^{\theta}},$$

with

$$c = -\frac{a}{\langle\tau\rangle} \cos(\pi\theta/2) = \frac{\pi\,\tau_0^{\theta}}{2\Gamma(\theta)\sin(\pi\theta/2)\langle\tau\rangle}.$$

In other words, we have the scaling $s \sim |u|^{\theta}$, or, for the typical value of S_t,

$$(S_t)_{\mathrm{typ}} \sim t^{1/\theta}.$$

The distribution of S_t is then given, for y small and t large, by the double inverse Laplace transform

$$f_S(t; y) \approx \int \frac{ds}{2\pi\mathrm{i}} \, \mathrm{e}^{st} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\omega y}}{s + c\,|\omega|^{\theta}} \approx \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, \mathrm{e}^{\mathrm{i}\omega y - c|\omega|^{\theta} t}.$$

Setting

$$S_t = (ct)^{1/\theta} X_t, \tag{7.14}$$

we conclude that the scaling variable X_t has a nontrivial even limiting distribution, as $t \to \infty$, with argument $x = y/(ct)^{1/\theta}$,

$$f_X(x) = L_\theta(x) = \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} e^{i\alpha x - |\alpha|^\theta}. \tag{7.15}$$

This distribution is the symmetric stable Lévy law L_θ of index θ. Expanding the integrand in the right side as a Taylor series in x yields the convergent series

$$L_\theta(x) = \frac{1}{\pi\theta} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma((2k+1)/\theta)}{(2k)!} x^{2k}.$$

For large values of x, $L_\theta(x)$ falls off as a power law,

$$L_\theta(x) \underset{x \to \pm\infty}{\approx} \frac{\Gamma(\theta+1) \sin(\pi\theta/2)}{\pi} |x|^{-\theta-1},$$

as obtained by expanding $e^{-|\alpha|^\theta}$ as $1 - |\alpha|^\theta$ in equation (7.15). As a consequence, the second moment of this distribution is divergent.

To summarize, the bulk distribution of S_t is given by $L_\theta(x)$, with the scaling (7.14), while the moments of S_t scale as (7.13). These two kinds of behavior are actually related, as shown by the following simple reasoning. Using (7.14), we have

$$\langle S_t^2 \rangle \sim t^{2/\theta} \langle X_t^2 \rangle \sim t^{2/\theta} \int_{-x_c}^{x_c} dx\, x^2\, L_\theta(x) \sim t^{2/\theta} x_c^{2-\theta}, \tag{7.16}$$

where x_c is an estimate for x in the tails. Since $x_c \sim S_c/t^{1/\theta}$ with $S_c \sim t$, by definition (1.2) of S_t, we finally obtain $\langle S_t^2 \rangle \sim t^{3-\theta}$ as in (7.13). Note that, since $2/\theta < 3 - \theta$, the typical value of S_t^2 is much smaller than its average, at large times. A similar situation arises in a model considered in ref. 27. We are indebted to J.P. Bouchaud for pointing this to us.

This argument can be generalized to the calculation of non-integer moments of S_t, yielding

$$\langle |S_t|^p \rangle \sim t^{\gamma(p)},$$

with $\gamma(p) = p/\theta$ if $p \leqslant \theta$, and $\gamma(p) = p + 1 - \theta$ if $p \geqslant \theta$, a behavior characteristic of a bifractal distribution.

In the limits $\theta \to 1$ and $\theta \to 2$, L_θ becomes respectively a Cauchy and a Gaussian distribution,

$$L_\theta(x) \underset{\theta \to 1}{\to} \frac{1}{\pi(1+x^2)}, \qquad L_\theta(x) \underset{\theta \to 2}{\to} \frac{e^{-x^2/4}}{2\sqrt{\pi}}.$$

8. NUMBER OF RENEWALS BETWEEN TWO ARBITRARY TIMES

Consider the number of events $N(t, t+t') = N_{t+t'} - N_t$ occurring between t and $t+t'$. The probability distribution of this random variable is denoted by

$$p_n(t, t+t') = \mathscr{P}(N(t, t+t') = n).$$

The time of occurrence of the n-th event, counted from time t, is denoted by t'_n, with, by convention, $t'_0 = 0$. By definition of the forward recurrence time E_t, the first event after time t occurs at time $t'_1 = E_t$, when counted from time t. Hence the time of occurrence of the last event before $t + t'$, counted from t, reads

$$t'_{N(t, t+t')} = E_t + \tau_2 + \cdots + \tau_{N(t, t+t')}.$$

Therefore,

$$p_n(t, t+t') = \mathscr{P}(t'_n < t' < t'_{n+1}) = \langle I(t'_n < t' < t'_{n+1}) \rangle \qquad (n \geqslant 0). \tag{8.1}$$

In particular, for $n = 0$, we have

$$p_0(t, t+t') = \mathscr{P}(E_t > t') = \int_{t'}^{\infty} dy \, f_E(t; y), \tag{8.2}$$

which is the persistence probability up to time $t + t'$, counted from the waiting time t.

In Laplace space, where u is conjugate to t', we thus obtain

$$\underset{t'}{\mathscr{L}} \, p_n(t, t+t') = \hat{p}_n(t, u) = \hat{f}_E(t; u) \, \hat{\rho}(u)^{n-1} \frac{1 - \hat{\rho}(u)}{u} \qquad (n \geqslant 1),$$

$$\underset{t'}{\mathscr{L}} \, p_0(t, t+t') = \hat{p}_0(t, u) = \frac{1 - \hat{f}_E(t; u)}{u},$$

$$\tag{8.3}$$

where $\hat{f}_E(t; u)$ is the Laplace transform of $f_E(t; y)$ with respect to y.

We discuss the results above according to the nature of the distribution of intervals $\rho(\tau)$.

(i) *Narrow distributions of intervals*

As seen in section 6, the renewal process reaches equilibrium at long times, with, for the forward recurrence time,

$$\hat{f}_{E,\,\mathrm{eq}}(u) = \frac{1 - \hat{\rho}(u)}{\langle \tau \rangle u}, \qquad f_{E,\,\mathrm{eq}}(y) = \frac{1}{\langle \tau \rangle} \int_y^\infty d\tau\, \rho(\tau).$$

Therefore the equilibrium distribution of the random variable $N(t, t+t')$ no longer depends on t. In particular, the average of $N(t, t+t')$ is equal to $t'/\langle \tau \rangle$.

For an exponential distribution of time intervals, we have

$$p_n(t, t+t') = \mathrm{e}^{-\lambda t'} \frac{(\lambda t')^n}{n!} \qquad (n \geqslant 0),$$

which is independent of t, showing that the Poisson point process is at equilibrium at all times.

(ii) *Broad distributions of intervals with index $\theta < 1$*

We restrict the discussion to the probability $p_0(t, t+t')$. In the scaling regime where t and t' are large and comparable, we have, according to equations (8.2) and (6.6),

$$
\begin{aligned}
p_0(t, t+t') &\approx \int_{t'/t}^\infty dx \lim_{t \to \infty} f_{t^{-1}E}(x) = \frac{\sin \pi\theta}{\pi} \int_{t'/t}^\infty dx \frac{1}{x^\theta(1+x)} \\
&= \int_0^{t/(t+t')} dx\, \beta_{\theta,\,1-\theta}(x) = g_1\!\left(\frac{t}{t+t'}\right).
\end{aligned}
\tag{8.4}
$$

In particular, in the regime of large separations between t and $t+t'$, we obtain the aging form of the persistence probability:

$$p_0(t, t+t') \underset{1 \ll t \ll t'}{\approx} \frac{\sin \pi\theta}{\pi\theta} \left(\frac{t'}{t}\right)^{-\theta}.$$

For example, for $\theta = \frac{1}{2}$,

$$p_0(t, t+t') = \frac{2}{\pi} \arctan \sqrt{\frac{t}{t'}} = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{t+t'}}.
\tag{8.5}$$

One can also compute

$$\mathop{\mathscr{L}}_{t,\,t'} \langle N(t, t+t')\rangle = \sum_0^\infty n\, \hat{p}_n(s, u) = \frac{\hat{\rho}(s) - \hat{\rho}(u)}{u(u-s)(1 - \hat{\rho}(u))(1 - \hat{\rho}(s))},$$

or alternatively use (3.6), to find, in the same regime ($1 \ll t \sim t'$),

$$\langle N(t, t+t')\rangle \approx \frac{\sin \pi\theta}{\pi\theta} \frac{(t+t')^\theta - t^\theta}{\tau_0^\theta}. \tag{8.6}$$

A consequence of (8.4) and (8.6) is that, for $1 \ll t' \ll t$, the probability of finding an event between t and $t+t'$ goes to zero. In other words, in order to have a chance to observe a renewal, one has to wait a duration t' of order t. The intuitive explanation is that, as t is growing, larger and larger intervals of time τ may appear. The density of events at large times is therefore decreasing.

(iii) *Broad distributions of intervals with index* $1 < \theta < 2$
First, equations (6.7) and (8.2) imply that, for $1 \ll t' \ll t$,

$$p_{0,\,\mathrm{eq}}(t, t+t') \approx \frac{\tau_0^\theta}{(\theta-1)\langle\tau\rangle} t'^{-(\theta-1)}. \tag{8.7}$$

Then, in the scaling regime where t and t' are large and comparable, we obtain, using equations (6.2) and (8.3),

$$\hat{p}_0(s, u) \approx \frac{a}{\langle\tau\rangle} \frac{s^{\theta-1} - u^{\theta-1}}{s(s-u)},$$

which by inversion yields

$$p_0(t, t+t') = \frac{\tau_0^\theta}{(\theta-1)\langle\tau\rangle} (t'^{-(\theta-1)} - (t+t')^{-(\theta-1)}). \tag{8.8}$$

For $1 \ll t' \ll t$ we recover (8.7), while for $1 \ll t \ll t'$ we obtain

$$p_0(t, t+t') \mathop{\approx}_{1 \ll t \ll t'} \frac{\tau_0^\theta t}{\langle\tau\rangle} t'^{-\theta}.$$

Equation (8.8) can be rewritten in scaling form as

$$p_0(t, t+t') = \frac{\tau_0^\theta}{(\theta-1)\langle\tau\rangle} t^{-(\theta-1)} g_2\!\left(\frac{t}{t+t'}\right), \tag{8.9}$$

with

$$g_2(x) = x^{\theta-1}((1-x)^{-(\theta-1)} - 1). \tag{8.10}$$

9. TWO-TIME AUTOCORRELATION FUNCTION

The two-time autocorrelation function of the σ-process reads

$$C(t, t+t') = \langle \sigma_t \sigma_{t+t'} \rangle = \sum_{n=0}^{\infty} (-1)^n p_n(t, t').$$

In Laplace space, using (8.3), one gets

$$\mathcal{L}_{t'} C(t, t+t') = \hat{C}(t, u) = \frac{1 - \hat{f}_E(t; u)}{u} - \hat{f}_E(t; u) \frac{1 - \hat{\rho}(u)}{u(1 + \hat{\rho}(u))}$$

$$= \frac{1}{u} \left(1 - \hat{f}_E(t; u) \frac{2}{1 + \hat{\rho}(u)} \right). \tag{9.1}$$

(i) *Narrow distributions of intervals*
At equilibrium, using (6.3), we have

$$\hat{C}_{eq}(u) = \frac{1}{u} \left(1 - \frac{2(1 - \hat{\rho}(u)(}{\langle \tau \rangle u(1 + \hat{\rho}(u))} \right).$$

Expanding the right side as a Taylor series in u yields the sum rules

$$\int_0^{\infty} dt' C_{eq}(t') = \frac{\langle \tau^2 \rangle - \langle \tau \rangle^2}{2 \langle \tau \rangle}, \qquad \int_0^{\infty} dt' t' \, C_{eq}(t') = \frac{\langle \tau^3 \rangle}{6 \langle \tau \rangle} - \frac{\langle \tau^2 \rangle}{2} + \frac{\langle \tau \rangle^2}{4}. \tag{9.2}$$

Since the second expression can be either positive or negative, depending on the detailed form of the distribution $\rho(\tau)$, the equilibrium correlation function $C_{eq}(t')$ is neither positive nor monotonic in general.

For an exponential distribution ρ, one has

$$C(t, t+t') = e^{-2\lambda t'}$$

which is independent of t, reflecting once again the fact that the process is at equilibrium at all times.

(ii) *Broad distributions of intervals*

In the regime where $s \sim u \ll 1$, the first term in the first line of equation (9.1) dominates upon the second one for any $0 < \theta < 2$, so that we have

$$C(t, t+t') \approx p_0(t, t+t'),$$

with $p_0(t, t+t')$ given by (8.4) for $\theta < 1$, and by (8.8) or (8.9) for $1 < \theta < 2$.

Let us finally remark that, from the knowledge of the two-time autocorrelation $C(t, t+t')$, we can recover the asymptotic behavior of $\langle S_t^2 \rangle$, or of $\langle M_t^2 \rangle$, respectively given by equations (7.3), (7.8) and (7.13), according to the nature of the distribution of intervals ρ. Indeed, by definition of S_t (see equation (1.2)), we have

$$\langle S_t^2 \rangle = \int_0^t dt_2 \int_0^t dt_1 \, C(t_1, t_2).$$

For a narrow distribution ρ, this yields, as $t \to \infty$,

$$\langle S_t^2 \rangle \approx 2t \int_0^\infty dt' \, C_{\text{eq}}(t'),$$

which, using (9.2) leads to (7.3). For a broad distribution ρ with $\theta < 1$, we have, in the long-time scaling regime

$$\langle S_t^2 \rangle \approx 2t^2 \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \, C\left(\frac{\tau_1}{\tau_2}\right) = t^2 \int_0^1 dx \, g_1(x), \tag{9.3}$$

where $g_1(x)$ is given in (8.4). A simple calculation then leads to $\langle M^2 \rangle$ as in (7.8). For a broad distribution ρ with $1 < \theta < 2$, using (8.9), we find

$$\langle S_t^2 \rangle \approx t^{3-\theta} \frac{2\tau_0^\theta}{(\theta-1)(3-\theta)\langle \tau \rangle} \int_0^1 dx \, x^{-(\theta-1)} \, g_2(x),$$

yielding, after integration, the first line of (7.13).

10. SUMMARY AND FINAL REMARKS

This article is devoted to the study of the occupation times $T_t^\pm$, and mean magnetization M_t, of renewal processes, that is, processes with independent intervals of time between events, interpreted as the zero crossings of the stochastic process $\sigma_t = \pm 1$. We also compute the distributions of the

random variables naturally associated to a renewal process, such that the number of events N_t occurring between 0 and t, the epoch t_N of the last event before t, the backward and forward times B_t and E_t. We finally investigate the number of events occurring between two arbitrary instants of time, and the two-time autocorrelation function.

The present work is also instructive for the understanding of the role of correlations on the behavior of the distributions of sums of random variables. Here the random variables are the signs $\sigma_t = \pm 1$, and

$$S_t \equiv t\, M_t = \int_0^t dt'\, \sigma_{t'} \tag{10.1}$$

is therefore the sum of temporally correlated random variables.

Three cases are to be considered for the discussion of the results, according to the nature of the distribution $\rho(\tau)$ of the intervals of time between the renewal events.

The case where ρ is narrow corresponds to the domain of application of the central limit theorem. Correlations between values of the sign process σ_t at two different instants of time are short-ranged. All the observables of interest mentioned above have narrow distributions, as well. In particular,

$$(S_t)_{\text{typ}} \sim t^{1/2},$$

and the limiting distribution of $S_t/t^{1/2}$ is Gaussian. The persistence probability $p_0(t)$, i.e., the probability that no event occurred up to time t, decreases, as $t \to \infty$, faster than a power law.

The case where ρ is broad, with index $\theta < 1$, corresponds to a maximum violation of the central limit theorem. The particular case where $\theta = \frac{1}{2}$ accounts for Brownian motion. Correlations are long-ranged:

$$\langle \sigma_t \sigma_{t+t'} \rangle \sim g_1\!\left(\frac{t}{t+t'} \right), \tag{10.2}$$

where the scaling function behaves as $(t'/t)^{-\theta}$ in the regime of large separation ($1 \ll t \ll t'$). The number N_t of events between 0 and t scales as t^θ. The random variables t_N, B_t, E_t, $T_t^{\pm}$ and S_t, have limiting distributions, as $t \to \infty$, once rescaled by t. In particular, focusing on S_t, the scaling behavior (10.2) implies that

$$(S_t)_{\text{typ}} \sim t. \tag{10.3}$$

The limiting distribution of $S_t/t \equiv M_t$ is given by equation (7.6). It is a U-shaped curve for $\theta < \theta_c \approx 0.59$, while its maximum is at zero, for $\theta > \theta_c$. It is singular at $M_t = \pm 1$, with an exponent equal to $\theta - 1$. The probability, $p_0(t)$, that no event occurred up to time t, decreases as $t^{-\theta}$, while the probability, $p_0(t, t+t')$, that no event occurred between t and $t+t'$, behaves as $(t'/t)^{-\theta}$ in the regime $1 \ll t \ll t'$. (We remind that in the long-time scaling regime, $\langle \sigma_t \sigma_{t+t'} \rangle \approx p_0(t, t+t')$.)

The case where ρ is broad, with index $1 < \theta < 2$, is intermediate. Correlations are again long-ranged:

$$\langle \sigma_t \sigma_{t+t'} \rangle \sim t^{-(\theta-1)} g_2\left(\frac{t}{t+t'}\right), \tag{10.4}$$

where the scaling function behaves as $(t'/t)^{-\theta}$ in the regime of large separation ($1 \ll t \ll t'$). The law of N_t, centered at $t/\langle \tau \rangle$, and rescaled by $t^{1/\theta}$, is broad, with index θ. As $t \to \infty$, B_t and E_t have equilibrium distributions, which are broad, with index $\theta - 1$. The scaling behavior (10.4) implies that

$$\langle S_t^2 \rangle \sim t^{3-\theta}, \tag{10.5}$$

while a more complete analysis leads to

$$(S_t)_{\text{typ}} \sim t^{1/\theta}, \tag{10.6}$$

and shows that the limiting distribution of $S_t/t^{1/\theta}$ is the symmetric stable Lévy law of index θ. The existence of the two scales (10.5) and (10.6) is also reflected in the bifractality of the distribution of S_t, in the sense that the non-integer moments scale as

$$\langle |S_t|^p \rangle \sim t^{\gamma(p)},$$

with $\gamma(p) = p/\theta$, if $p < \theta$, and $\gamma(p) = p + 1 - \theta$, if $p > \theta$. Otherwise stated, there is no "gap scaling" for the distribution of S_t (see below). The behavior of $p_0(t)$ is the same as for the previous case $\theta < 1$. However, $p_0(t, t+t')$, though still decaying as $t'^{-\theta}$ in the regime $1 \ll t \ll t'$, is no longer given by a scaling function of the ratio $t/t+t'$, as was the case when $\theta < 1$ (compare equations (8.4) and (8.9)).

The three types of behavior summarized above, corresponding to a narrow distribution ρ (i.e., formally $\theta > 2$), to a broad distribution with $\theta < 1$, and to a broad distribution with $1 < \theta < 2$, are reminiscent of the three typical behaviors observed in the nonequilibrium dynamics of phase

ordering, respectively at high temperature, at low temperature, and at criticality. Consider for instance a two-dimensional lattice of Ising spins $\sigma_t = \pm 1$. At high temperature, correlations between spins are short-ranged, and the analysis given above applies, i.e., the central limit theorem is obeyed by S_t. For a quench from a disordered initial state to a temperature below the critical temperature, the autocorrelation has a form similar to (10.2), namely

$$\langle \sigma_t \sigma_{t+t'} \rangle \sim m_{\text{eq}}^2\, g_1^{\text{Ising}}\!\left(\frac{t}{t+t'}\right),$$

where m_{eq} is the equilibrium magnetization. The scaling function g_1^{Ising} behaves as $(t'/t)^{-\lambda/z}$ for $1 \ll t \ll t'$, where $z = 2$ is the growth exponent, and $\lambda \approx 1.25$ is the autocorrelation exponent. Correspondingly (10.3) holds. The limiting distribution of $S_t/t \equiv M_t$ is a U-shaped curve, with singularity exponent, as $M_t \to \pm m_{\text{eq}}$, equal to $\theta - 1$, where $\theta \approx 0.22$ is the persistence exponent at low temperature. [12]

Finally for a quench at the critical temperature, correlations have a form similar to (10.4),

$$\langle \sigma_t \sigma_{t+t'} \rangle \sim t^{-2\beta/\nu z_c} g_2^{\text{Ising}}\!\left(\frac{t}{t+t'}\right), \tag{10.7}$$

where $\beta = 1/8$ and $\nu = 1$ are the usual static critical exponents, and $z_c \approx 2.17$ is the dynamic critical exponent. The scaling function g_2^{Ising} behaves as $(t'/t)^{-\lambda_c/z_c}$ for $1 \ll t \ll t'$, where $\lambda_c \approx 1.59$ is the critical autocorrelation exponent (see e.g. ref. 18). However, in contrast with the situation encountered in the present work, one expects, by analogy with statics, that the distribution of S_t is entirely described by one scale, given by $(S_t)_{\text{typ}} = \langle S_t^2 \rangle^{\frac{1}{2}} \sim t^{1-\beta/\nu z_c}$. In this sense, this distribution is a monofractal, and gap scaling holds: all moments scale as $\langle |S_t|^p \rangle \sim (S_t)_{\text{typ}}^p \sim t^{\gamma(p)}$, with $\gamma(p) = p(1 - \beta/\nu z_c)$. These expectations are confirmed by numerical computations. [28]

Pursuing the comparison between the two situations, persistence for coarsening systems decays faster than a power law at high temperature, and as a power law at low temperature, as is the case for a renewal process, respectively for ρ narrow, and for ρ broad, with $\theta < 1$. Again the status of the intermediate case is different: while for the renewal processes under study $p_0(t)$ is still decaying as $t^{-\theta}$ when $1 < \theta < 2$, for critical coarsening the decay of the probability that no spin flip occurred up to time t, $p_0(t)$, is no longer algebraic.

To conclude, as we already mentioned, the methods introduced here provide a basis for the study of similar questions for the stochastic process

considered in ref. 25, and to some extent, for the random acceleration problem.[29]

After completion of this article, we became aware of recent works in probability theory on various quantities derived from Brownian motion, whose distributions are proven to be beta laws, generalizing thus Lévy's arcsine law.[30]

A. A WORD ON NOTATIONS

Probability densities

Consider the time-dependent random variable Y_t. We are interested in the distribution function of this random variable, $\mathscr{P}(Y_t < y)$, and in its probability density function, denoted by f_Y, dropping the time dependence of the random variable when it is in subscript,

$$f_Y(t; y) = \frac{d}{dy} \mathscr{P}(Y_t < y).$$

Time t appears as a parameter in this function.

Laplace transforms

Assume that Y_t is positive. We denote the Laplace transform of $f_Y(t; y)$ with respect to y as

$$\mathscr{L}_y f_Y(t; y) = \hat{f}_Y(t; u) = \langle e^{-uY_t} \rangle = \int_0^\infty dy\, e^{-uy} f_Y(t; y),$$

and the double Laplace transform of $f_Y(t; y)$ with respect to t and y as

$$\mathscr{L}_{t, y} f_Y(t; y) = \mathscr{L}_t \langle e^{-uY_t} \rangle = \hat{f}_Y(s; u) = \int_0^\infty dt\, e^{-st} \int_0^\infty dy\, e^{-uy} f_Y(t; y).$$

In this work we encounter random variables Y_t (such as S_t or M_t) with even distributions on the real axis, i.e., $f_Y(t; y) = f_Y(t; -y)$. For these we define the (bilateral) Laplace transform as

$$\mathscr{L}_y f_Y(t; y) = \langle e^{-uY_t} \rangle = \int_{-\infty}^\infty dy\, e^{-uy} f_Y(t; y).$$

Limiting distributions

In a number of instances considered in this work, Y_t scales asymptotically as t. Therefore, it is natural to define the scaling variable $X_t = Y_t/t$,

with density $f_X(t; x) \equiv f_{t^{-1}Y}(t; x = y/t)$. As $t \to \infty$, this density converges to a limit, denoted by

$$f_X(x) = \lim_{t \to \infty} f_{t^{-1}Y}(x).$$

These considerations hold similarly for other scaling forms of Y_t.

B. INVERSION OF THE SCALING FORM OF A DOUBLE LAPLACE TRANSFORM

Consider the probability density function $f_Y(t; y)$ of the random variable Y_t, and assume that its double Laplace transform with respect to t and y, defined in appendix A, has the scaling behavior

$$\mathop{\mathscr{L}}_{t, y} f_Y(t; y) = \hat{f}_Y(s; u) = \frac{1}{s} g\left(\frac{u}{s}\right) \tag{B.1}$$

in the regime $s, u \to 0$, with u/s arbitrary. Then the following properties hold, as shown below.

(i) The random variable $X_t = Y_t/t$ possesses a limiting distribution when $t \to \infty$, i.e.,

$$f_X(x) = \lim_{t \to \infty} f_{t^{-1}Y}(t; x = y/t). \tag{B.2}$$

(ii) The scaling function g is related to f_X by

$$g(\xi) = \left\langle \frac{1}{1 + \xi X} \right\rangle = \int_{-\infty}^{\infty} dx \frac{f_X(x)}{1 + \xi x}. \tag{B.3}$$

(iii) This can be inverted as

$$f_X(x) = -\frac{1}{\pi x} \lim_{\epsilon \to 0} \operatorname{Im} g\left(-\frac{1}{x + i\epsilon}\right). \tag{B.4}$$

(iv) Finally the moments of f_X can be obtained by expanding $g(y)$ as a Taylor series, since (B.3) implies that

$$g(\xi) = \sum_{k=0}^{\infty} (-\xi)^k \langle X^k \rangle. \tag{B.5}$$

First, a direct consequence of the scaling form (B.1) is that Y_t scales as t, as can be seen by Taylor expanding the right side of this equation, which generates the moments of Y_t in the Laplace space conjugate to t. Therefore (B.2) holds.

Then, (B.3) is a simple consequence of (B.2), since

$$\hat{f}_Y(s; u) = \int_0^\infty dt \; e^{-st} \langle e^{-uY} \rangle = \int_0^\infty dt \, e^{-st} \langle e^{-ut\,X} \rangle = \left\langle \frac{1}{s + uX} \right\rangle.$$

Now,

$$f_X(x) = \langle \delta(x - X) \rangle = -\frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im} \left\langle \frac{1}{x + i\epsilon - X} \right\rangle.$$

The right side can be rewritten using (B.3), yielding (B.4).

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